Sonine Transform Associated to the Bessel-Struve Operator *

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Abstract

In this paper we consider the Bessel-Struve operator l_α and the Bessel-Struve intertwining operator χ_α and its dual, we define and study the Bessel-Struve Sonine transform $S_{\alpha,\beta}$ on $\mathcal{E}(\mathbb{R})$. We prove that $S_{\alpha,\beta}$ is a transmutation operator from l_α into l_β on $\mathcal{E}(\mathbb{R})$ and we deduce similar result for its dual $S^*_{\alpha,\beta}$ on $\mathcal{E}'(\mathbb{R})$. Furthermore, invoking Weyl integral transform and the Dual Sonine transform ${}^tS_{\alpha,\beta}$ on $\mathcal{D}(\mathbb{R})$, we get a relation between the Bessel-Struve transforms $\mathcal{F}^\alpha_{BS}$ and $\mathcal{F}^\beta_{BS}$.

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1 Introduction

Among the formulae listed in Watson's classical monograph [9] there is the following one due to Sonine [6]

$$j_\alpha(\lambda\, x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_0^1 (1-t^2)^{\alpha-\beta-1} j_\beta(\lambda\, x\, t)\, t^{2\beta+1}\, dt$$

where j_α is the normalized Bessel function

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)z^{-\alpha}J_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\alpha+1)}$$

K.Trimèche introduced the Sonine integral transform, in [8], by

$$\forall x \geq 0, \quad S_{\alpha,\beta}(f)(x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_0^1 (1-r^2)^{\alpha-\beta-1} f(rx)\, r^{2\beta+1} dr$$

This transform is related to the Bessel operator defined by

$$L_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\left(\frac{du}{dx}(x)\right) \ , \quad x \in \mathbb{R}$$

where u designates an even function infinitely differentiable on $\mathbb{R}$.
In this paper, we consider the differential operator l_α , $\alpha > -\frac{1}{2}$, defined by

$$l_\alpha u(x) = \frac{d^2u}{dx^2}(x) + \frac{2\alpha+1}{x}\left(\frac{du}{dx}(x) - \frac{du}{dx}(0)\right), \quad x \in \mathbb{R} \qquad (1)$$

with u an infinitely differentiable function on $\mathbb{R}$.
This operator is called Bessel-Struve operator. We remark that l_α can be expressed in the following form

$$\forall\, x \in \mathbb{R}^*, \qquad l_\alpha(u)(x) = \frac{1}{|x|^{2\alpha+1}}\frac{d}{dx}\left(|x|^{2\alpha+1}\left(u'(x) - u'(0)\right)\right) \qquad (2)$$

For $\lambda \in \mathbb{C}$, the differential equation :

$$\begin{cases} l_\alpha u = \lambda^2\, u \\ u(0) = 1, u'(0) = \dfrac{\lambda\,\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} \end{cases}$$

possesses a unique solution denoted $\Phi_\alpha(\lambda.)$ and called Bessel-Struve kernel. This kernel possesses the following integral representation :

$$\forall x \in \mathbb{R},\ \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{\lambda x t}\, dt \qquad (3)$$

Φ_α can be expressed using normalized Bessel function j_α and normalized Struve function h_α by :

$$\Phi_\alpha(x) = j_\alpha(ix) - i\, h_\alpha(ix)\ , \qquad x \in \mathbb{R} \qquad (4)$$

where

$$h_\alpha(z) = 2^\alpha \Gamma(\alpha+1) z^{-\alpha} \mathbf{H}_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{3}{2})}$$

Therefore, the Bessel-Struve kernel can be extended to an analytic function on $\mathbb{C}$ and it has the form in a power series :

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{z^n}{c_n(\alpha)}\ , \qquad z \in \mathbb{C} \qquad (5)$$

where $\quad c_n(\alpha) = \dfrac{\sqrt{\pi}\, n!\, \Gamma(\frac{n}{2}+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\frac{n+1}{2})}$

The outline of the content of the paper is as follows
In section 2, we give some results about harmonic analysis associated to Bessel-Struve operator which we will use later.
In section 3, we deal with Sonine transform defined by

$$\forall x \in \mathbb{R}, \quad S_{\alpha,\beta}(f)(x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^1 (1-r^2)^{\alpha-\beta-1} f(rx)\, r^{2\beta+1} dr$$

Firstly, we find that $S_{\alpha,\beta}$ verifies the following relation

$$\forall f \in \mathcal{E}(\mathbb{R}), \quad l_\alpha(S_{\alpha,\beta}(f)) = S_{\alpha,\beta}(l_\beta(f))$$

Next, we prove that $S_{\alpha,\beta}$ is an isomorphism from $\mathcal{E}(\mathbb{R})$ into it self. These two statements allows us to say that $S_{\alpha,\beta}$ is a transmutation operator from l_α into l_β on $\mathcal{E}(\mathbb{R})$.
In section 4, we define the dual of $S_{\alpha,\beta}$ on $\mathcal{E}'(\mathbb{R})$ denoted $S^*_{\alpha,\beta}$. We prove that

$S^*_{\alpha,\beta}$ is a transmutation operator from l_β into l_α on $\mathcal{E}'(\mathbb{R})$. Furthermore, we consider the dual Sonine integral transform ${}^tS_{\alpha,\beta}$ in the following sense

$$\int_{\mathbb{R}} S_{\alpha,\beta}(g)(x)\, f(x)\, |x|^{2\alpha+1}\, dx \;=\; \int_{\mathbb{R}} {}^tS_{\alpha,\beta}(f)(x)\, g(x)\, |x|^{2\beta+1}\, dx$$

We express ${}^tS_{\alpha,\beta}$ using Weyl integral transform associated to Bessel-Struve operator introduced by the authors in [1] and we find a relation between the Bessel-Struve transforms $\mathcal{F}^\alpha_{B,S}$ and $\mathcal{F}^\beta_{B,S}$ on $\mathcal{D}(\mathbb{R})$

Similar results about Sonine transform have been obtained by K.Trimèche in [8] for the Bessel operator. Recently, Mourou in [3], [4] and Soltani in [7] obtain analogous results in the framework of Dunkl operator.

2 Bessel-Struve intertwining operator and its dual

$\mathcal{E}(\mathbb{R})$ designates the space of infinitely differentiable functions f on $\mathbb{R}$, provided with the topology defined by the semi norms

$$p_{n,a}(f) = \sup_{\substack{0\leq k\leq n\\ x\in[-a,a]}} |f^{(k)}(x)|$$

where $a > 0$ and $n \in \mathbb{N}$.
The Bessel-Struve intertwining operator on $\mathbb{R}$ denoted χ_α, introduced by L. Kamoun and M. Sifi in [2], is defined by :

$$\chi_\alpha(f)(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} f(xt)\, dt \quad , \; f \in \mathcal{E}(\mathbb{R}),\; x \in \mathbb{R} \tag{6}$$

It verifies the following properties
Let f be in $\mathcal{E}(\mathbb{R})$

$$\chi_\alpha(f)(0) = f(0) \tag{7}$$

$$[\chi_\alpha(f)]'(0) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} f'(0) \tag{8}$$

$$\forall\, \lambda \in \mathbb{R}, \qquad \Phi_\alpha(\lambda\, .) \;=\; \chi_\alpha(e^{\lambda\, \cdot}) \tag{9}$$

We consider the differential operator $\dfrac{d}{dx^2} = \dfrac{1}{2x}\dfrac{d}{dx}$.

Theorem 2.1 *[2, Theorem 1] The operator χ_α, $\alpha > -\frac{1}{2}$, is a topological isomorphism from $\mathcal{E}(\mathbb{R})$ onto itself. The inverse operator χ_α^{-1} is given for all f in $\mathcal{E}(\mathbb{R})$ by*

(i) *if $\alpha = r + k$, $k \in \mathbb{N}$, $-\frac{1}{2} < r < \frac{1}{2}$*

$$\chi_\alpha^{-1} f(x) = \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}-r)} x(\frac{d}{dx^2})^{k+1} \left[\int_0^x (x^2-t^2)^{-r-\frac{1}{2}} |t|^{2\alpha+1} f(t)\, dt\right]$$

(ii) *if $\alpha = \frac{1}{2} + k$, $k \in \mathbb{N}$*

$$\chi_\alpha^{-1} f(x) = \frac{2^{2k+1} k!}{(2k+1)!}\, x\,(\frac{d}{dx^2})^{k+1}\big(x^{2k+1} f(x)\big), \quad x \in \mathbb{R}$$

Proposition 2.1 *[2, Proposition 1] χ_α is a transmutation operator from l_α into D^2 on $\mathcal{E}(\mathbb{R})$. Namely χ_α is an isomorphism from $\mathcal{E}(\mathbb{R})$ into itself and verifies*

$$l_\alpha \circ \chi_\alpha = \chi_\alpha \circ D^2$$

Definition 2.1 *We define the dual transform of χ_α, denoted χ_α^*, on $\mathcal{E}'(\mathbb{R})$ by*

$$< \chi_\alpha^*(T), f > = < T, \chi_\alpha f > , \qquad T \in \mathcal{E}'(\mathbb{R}) , \quad f \in \mathcal{E}(\mathbb{R}) \tag{10}$$

Theorem 2.2 *[1, Corrollary 3.1] χ_α^* is an isomorphism from $\mathcal{E}'(\mathbb{R})$ into itself.*

We denote by $L^1_\alpha(\mathbb{R})$ the space of measurable functions f verifying

$$\int_{\mathbb{R}} |f(t)|\, |t|^{2\alpha+1}\, dt \ < \ +\infty$$

Definition 2.2 *For $f \in L^1_\alpha(\mathbb{R})$ with bounded support, we define the integral transform W_α by*

$$W_\alpha f(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_{|y|}^{+\infty} (x^2-y^2)^{\alpha-\frac{1}{2}}\, x\, f(sgn(y)x)\, dx\,, \quad y \in \mathbb{R}^\star \tag{11}$$

W_α is called Weyl integral transform associated to Bessel-Struve operator.

Remark 2.1 *By a change of variable, $W_\alpha f$ can be written*

$$W_\alpha f(y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}|y|^{2\alpha+1}\int_1^{+\infty}(t^2-1)^{\alpha-\frac{1}{2}}\,t\,f(ty)\,dt\,, \quad y\in\mathbb{R}^\star \tag{12}$$

We designate by $\mathcal{K}_0$ the space of functions f infinitely differentiable on $\mathbb{R}^*$ with bounded support verifying for all $n\in\mathbb{N}$,

$$\lim_{\substack{y\to 0\\ y>0}} y^n f^{(n)}(y) \quad \text{and} \quad \lim_{\substack{y\to 0\\ y<0}} y^n f^{(n)}(y)$$

exist.

Remark 2.2 *From Lemma 3.1 of [1], one can see that, if f belongs to $\mathcal{D}(\mathbb{R})$ then $W_\alpha(f)$ belongs to $\mathcal{K}_0$. On the other hand, Proposition 3.2 of [1] says that W_α is a bounded operator from $L^1_\alpha(\mathbb{R})$ into $L^1(\mathbb{R})$.*

Proposition 2.2 *Let f be a function in $\mathcal{E}(\mathbb{R})$ and g a function in $L^1_\alpha(\mathbb{R})$ with bounded support, the operators χ_α and W_α are related by the following relation*

$$\int_{\mathbb{R}}\chi_\alpha f(x)\,g(x)|x|^{2\alpha+1}\,dx \;=\; \int_{\mathbb{R}} f(x)\,W_\alpha g(x)dx \tag{13}$$

Proof. By a change of variable, we have

$$\int_{\mathbb{R}}\chi_\alpha f(x)\,g(x)|x|^{2\alpha+1}\,dx \;= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{\mathbb{R}}\left(\int_0^x(x^2-y^2)^{\alpha-\frac{1}{2}}f(y)dy\right)x\,g(x)dx$$

Using Chasles relation and applying Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\chi_\alpha f(x)\,g(x)|x|^{2\alpha+1}\,dx \;=&\; \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^{+\infty}\left(\int_y^{+\infty}(x^2-y^2)^{\alpha-\frac{1}{2}}x\,g(x)dx\right)f(y)dy\\ &-\frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-\infty}^0\left(\int_{-\infty}^y(x^2-y^2)^{\alpha-\frac{1}{2}}x\,g(-x)dx\right)f(y)dy\end{aligned}$$

Finally, by a change of variable, we get

$$\begin{aligned}\int_{\mathbb{R}}\chi_\alpha f(x)\,g(x)|x|^{2\alpha+1}\,dx &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{\mathbb{R}}\left(\int_{|y|}^{+\infty}(x^2-y^2)^{\alpha-\frac{1}{2}}x\,g((sign(y)x)dx\right)f(y)dy\\ &= \int_{\mathbb{R}} f(y)\,W_\alpha g(y)dy\end{aligned}$$

$\square$

Proposition 3.4 of [1] allows us to give the following definition

Definition 2.3 *We define the operator V_α on $\mathcal{K}_0$ as follows*

(i) *If $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$ and $f \in \mathcal{K}_0$*

$$V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1} k!}{(2k+1)!} \left(\frac{d}{dx^2}\right)^{k+1} (f(x)), \quad x \in \mathbb{R}^\star$$

(ii) *If $\alpha = k + r$, $k \in \mathbb{N}$, $-\frac{1}{2} < r < \frac{1}{2}$ and $f \in \mathcal{K}_0$*

$$V_\alpha f(x) = c_1 \int_{|x|}^{+\infty} (y^2 - x^2)^{-r-\frac{1}{2}} \left(\frac{d}{dy^2}\right)^{k+1} (f)(sgn(x)y)\, y\, dy\,, \;\; x \in \mathbb{R}^\star$$

$$\textit{where} \qquad c_1 = \frac{(-1)^{k+1} 2\sqrt{\pi}}{\Gamma(\alpha+1)\Gamma(\frac{1}{2} - r)}$$

Proposition 2.3 *For f in $\mathcal{K}_0$, $V_\alpha(f)$ belongs to $L^1_\alpha(\mathbb{R})$ and has a bounded support.*

Proof. If $supp(f)$ is included in $[-a, a]$ then it's clear that $supp(V_\alpha)$ is included in $[-a, a]$.
For $\alpha = k + \frac{1}{2}$, we obtain from [1, Lemma 3.2]

$$V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1} k!\, x}{(2k+1)!} \sum_{i=0}^{k+1} \beta_i^{k+1} x^{-2k-2+i} f^{(i)}(x)$$

Since f be in $\mathcal{K}_0$, we get $|x|^{2k+2} V_\alpha(f) \in L^1(\mathbb{R})$ which proves that $V_\alpha f$ belongs to $L^1_\alpha(\mathbb{R})$.
For $\alpha = k + r$, if $x > 0$, from [1, Lemma 3.2]

$$V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_x^{+\infty} (y^2 - x^2)^{-r-\frac{1}{2}} y^{-2k-1+i} f^{(i)}(y) dy$$

By a change of variables,

$$x^{2\alpha+1} V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_1^{+\infty} (t^2 - 1)^{-r-\frac{1}{2}} t^{-2k-1} (tx)^i f^{(i)}(tx) dt$$

If $x < 0$, by a change of variables and using [1, Lemma 3.2], we can write

$$V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} \int_{-\infty}^{x} (y^2 - x^2)^{-r-\frac{1}{2}} y^{-2k-1+i} f^{(i)}(y)dy$$

$$|x|^{2\alpha+1} V_\alpha f(x) = c_1 \sum_{i=0}^{k+1} \beta_i^{k+1} (sign(x))^{2k+1} \int_{1}^{+\infty} (t^2-1)^{-r-\frac{1}{2}} t^{-2k-1} (tx)^i f^{(i)}(tx)dt$$

Therefore, we see that $\lim\limits_{\substack{x\to 0\\ x>0}} |x|^{2\alpha+1} V_\alpha f(x)$ exists from dominated convergence theorem. Since $V_\alpha f$ is with bounded support then $V_\alpha f$ belongs to $L^1_\alpha(\mathbb{R})$ $\square$

Proposition 2.4 *[1, Remark 3.3] The operators V_α and χ_α^{-1} are related by the following relation*

$$\int_{\mathbb{R}} V_\alpha f(x)\, g(x) |x|^{2\alpha+1}\, dx \;=\; \int_{\mathbb{R}} f(x)\, \chi_\alpha^{-1} g(x)\, dx \tag{14}$$

for all $f \in \mathcal{K}_0$ and $g \in \mathcal{E}(\mathbb{R})$

Lemma 2.1 *Let f be a function in $\mathcal{K}_0$ then $W_\alpha(V_\alpha(f)) = f$ on $\mathbb{R}^*$*

Proof. Using Proposition 2.3 and relation (13), we get, for all $g \in \mathcal{E}(\mathbb{R})$

$$\int_{\mathbb{R}} W_\alpha(V_\alpha(f))(x)\, g(x)\, dx = \int_{\mathbb{R}} V_\alpha(f)(x)\, \chi_\alpha(g(x)) |x|^{2\alpha+1}\, dx$$

By relation(14), we deduce that

$$\int_{\mathbb{R}} W_\alpha(V_\alpha(f))(x)\, g(x)\, dx = \int_{\mathbb{R}} f(x)\, g(x)\, dx$$

Therefore

$$W_\alpha(V_\alpha(f))(x) = f(x)\ , \qquad a.e.\ x \in \mathbb{R}$$

Since $W_\alpha(V_\alpha(f))$ and f are both continuous functions on $\mathbb{R}^*$, we get for all $x \in \mathbb{R}^*$, $\qquad W_\alpha(V_\alpha(f))(x) = f(x)$. $\square$

3 Sonine integral transform

Throughout this section α and β are two real numbers verifying $\alpha > \beta > \dfrac{-1}{2}$.
In the next Proposition, we establish an analogue of Sonine formula

Proposition 3.1 *We have the following relation*

$$\Phi_\alpha(\lambda\, x) = a_{\alpha,\beta} \int_0^1 (1-t^2)^{\alpha-\beta-1} \Phi_\beta(\lambda\, x\, t)\, t^{2\beta+1}\, dt \tag{15}$$

where

$$a_{\alpha,\beta} = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}$$

Proof. From relation (5), we have

$$\int_0^1 (1-t^2)^{\alpha-\beta-1} \Phi_\beta(\lambda\, x\, t)\, t^{2\beta+1}\, dt = \sum_{n=0}^{+\infty} \frac{(\lambda\, x)^n}{c_n(\beta)} I_n(\alpha,\beta)$$

where

$$\begin{aligned} I_n(\alpha,\beta) &= \int_0^1 (1-t^2)^{\alpha-\beta-1}\, t^{2\beta+n+1} dt \\ &= \frac{1}{2} \int_0^1 (1-y)^{\alpha-\beta-1}\, y^{\beta+\frac{n}{2}} dy \\ &= \frac{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{n}{2}+1)}{2\,\Gamma(\alpha+\frac{n}{2}+1)} \end{aligned}$$

Then

$$\int_0^1 (1-t^2)^{\alpha-\beta-1} \Phi_\beta(\lambda\, x\, t)\, t^{2\beta+1}\, dt = \frac{\Gamma(\alpha-\beta)\Gamma(\beta+1)}{2\,\Gamma(\alpha+1)}\, \Phi_\alpha(\lambda\, x)$$

□

Definition 3.1 *Let f be a continuous function on $\mathbb{R}$. We define the Sonine integral transform as in [8] by, for all $x \in \mathbb{R}$*

$$S_{\alpha,\beta}(f)(x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^1 (1-r^2)^{\alpha-\beta-1} f(rx)\, r^{2\beta+1} dr \tag{16}$$

Remark 3.1 *The following relation yields from relation (15)*

$$S_{\alpha,\beta}(\Phi_\beta(\lambda\, .))(x) = \Phi_\alpha(\lambda\, x)\,, \qquad x \in \mathbb{R}$$

Proposition 3.2 *For f bounded continuous function on $\mathbb{R}$, the function $S_{\alpha,\beta}(f)$ is continuous on $\mathbb{R}$ and we have*

$$\|S_{\alpha,\beta}(f)\|_\infty \leq \|f\|_\infty \tag{17}$$

where $\quad \|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$

Proof. The result follows from continuity's theorem and the fact that

$$\int_0^1 (1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1} dr = \frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{2\Gamma(\alpha+1)}$$

□

Theorem 3.1 *For f a function of class C^2 on $\mathbb{R}$, $S_{\alpha,\beta}(f)$ is a function of class C^2 on $\mathbb{R}$ and we have*

$$\forall x \in \mathbb{R}, \quad l_\alpha(S_{\alpha,\beta}(f))(x) = S_{\alpha,\beta}(l_\beta(f))(x) \tag{18}$$

Proof. Using the theorem of derivation under the integral sign, we can prove that $S_{\alpha,\beta}(f)$ is of class C^2 on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \quad [S_{\alpha,\beta}(f)]'(x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^1 (1-r^2)^{\alpha-\beta-1} f'(rx)\, r^{2\beta+2} dr$$

By making the change of variable $t = rx$, we get for all $x \in \mathbb{R}^\star$
$[S_{\alpha,\beta}(f)]'(x) - [S_{\alpha,\beta}(f)]'(0) =$

$$\frac{2\Gamma(\alpha+1) sign(x) |x|^{-2\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha-\beta)} \int_0^x (x^2-t^2)^{\alpha-\beta-1} [f'(t) - f'(0)]\, |t|^{2\beta+2} dt$$

Next, invoking relation (2), we obtain by integration by parts

$$[S_{\alpha,\beta}(f)]'(x) - [S_{\alpha,\beta}(f)]'(0) = \frac{\Gamma(\alpha+1)|x|^{-2\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} \int_0^x (x^2-t^2)^{\alpha-\beta} l_\beta(f)(t)\, |t|^{2\beta+1} dt$$

we derive the two sides of the equation above, we obtain by virtue of the theorem of derivation under the integral sign

$$[S_{\alpha,\beta}(f)]''(x) = \frac{(2\alpha+1)\Gamma(\alpha+1)(-sign(x))|x|^{-2\alpha-2}}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} \int_0^x (x^2-t^2)^{\alpha-\beta} l_\beta(f)(t)\, |t|^{2\beta+1} dt$$

$$+\frac{\Gamma(\alpha+1)2x|x|^{-2\alpha-1}}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_0^x (x^2-t^2)^{\alpha-\beta-1} l_\beta(f)(t)\,|t|^{2\beta+1}dt$$

Then

$$\begin{aligned}
l_\alpha(S_{\alpha,\beta}(f))(x) &= [S_{\alpha,\beta}(f)]''(x)+\frac{2\alpha+1}{x}\big([S_{\alpha,\beta}(f)]'(x)-[S_{\alpha,\beta}(f)]'(0)\big)\\
&= \frac{2\Gamma(\alpha+1)sign(x)|x|^{-2\alpha}}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_0^x (x^2-t^2)^{\alpha-\beta-1} l_\beta(f)(t)\,|t|^{2\beta+1}dt\\
&= \frac{2\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta)}\int_0^1 (1-u^2)^{\alpha-\beta-1} l_\beta(f)(xu)\,u^{2\beta+1}du\\
&= S_{\alpha,\beta}(l_\beta f)(x)
\end{aligned}$$

□

Proposition 3.3 *For all f in $\mathcal{E}(\mathbb{R})$ the function $S_{\alpha,\beta}(f)$ belongs to $\mathcal{E}(\mathbb{R})$. The operator $S_{\alpha,\beta}$ is continuous from $\mathcal{E}(\mathbb{R})$ into itself.*

Proof. We deduce the wanted result using the theorem of derivation and proposition 3.2. □

Remark 3.2 *Let f be a function of class C^1 on $\mathbb{R}$, we have*

$$S_{\alpha,\beta}(f)(0) = f(0) \tag{19}$$

$$[S_{\alpha,\beta}(f)]'(0) = \frac{\Gamma(\alpha+1)\Gamma(\beta+\frac{3}{2})}{\Gamma(\beta+1)\Gamma(\alpha+\frac{3}{2})}f'(0) \tag{20}$$

Theorem 3.2 *The Sonine transform is a topological isomorphism from $\mathcal{E}(\mathbb{R})$ into itself. Furthermore, it verifies*

$$S_{\alpha,\beta} = \chi_\alpha \circ \chi_\beta^{-1} \tag{21}$$

The inverse operator is

$$S_{\alpha,\beta}^{-1} = \chi_\beta \circ \chi_\alpha^{-1} \tag{22}$$

Proof. We denote

$$P_{\alpha,\beta} = S_{\alpha,\beta}(f) - \chi_\alpha \circ \chi_\beta^{-1}$$

Using theorem 3.1, relation (9) and proposition 2.1, we have

$$\begin{aligned}
l_\alpha P_{\alpha,\beta}(\Phi_\beta(\lambda.)) &= S_{\alpha,\beta}\circ l_\beta \circ \chi_\beta(e^{\lambda\cdot}) - l_\alpha \circ \chi_\alpha(e^{\lambda\cdot})\\
&= \lambda^2\, S_{\alpha,\beta}\circ \chi_\beta(e^{\lambda\cdot}) - \lambda^2 \chi_\alpha(e^{\lambda\cdot})\\
&= \lambda^2\, P_{\alpha,\beta}(\Phi_\beta(\lambda\,.))
\end{aligned}$$

Furthermore, from relations (7) and (19), we deduce that

$$P_{\alpha,\beta}(\Phi_\beta(\lambda.))(0) = 0$$

and, from relations (8) and (20), we get

$$[P_{\alpha,\beta}(\Phi_\beta(\lambda.))]'(0) = 0$$

From the uniqueness of the solution of the differential equation $l_\alpha u = \lambda^2 u$ with the initial condition $u(0) = u'(0) = 0$, we obtain $P_{\alpha,\beta}(\Phi_\beta(\lambda\,.)) = 0$.
Thus the density of the family $\{\Phi_\alpha(\lambda\,.)\}_{\lambda\in\mathbb{R}}$ in $\mathcal{E}(\mathbb{R})$ implies that for all $f \in \mathcal{E}(\mathbb{R})$, $P_{\alpha,\beta}(f) = 0$ which proves the relation (21) □

Remark 3.3 *By theorem 3.1 and theorem 3.2, we conclude that $S_{\alpha,\beta}$ is a transmutation operator from l_α into l_β on $\mathcal{E}(\mathbb{R})$.*

4 The Dual Sonine transform

Definition 4.1 *Since l_α is a bounded linear operator from $\mathcal{E}(\mathbb{R})$ into itself, we define, for $T \in \mathcal{E}'(\mathbb{R})$, $l_\alpha T$ the compactly supported distribution on $\mathbb{R}$ by*

$$< l_\alpha T, f > = < T, l_\alpha f > , \quad f \in \mathcal{E}(\mathbb{R})$$

Theorem 4.1 *The dual transform $S^*_{\alpha,\beta}$ of $S_{\alpha,\beta}$ defined on $\mathcal{E}'(\mathbb{R})$ by*

$$< S^*_{\alpha,\beta} T, f > = < T, S_{\alpha,\beta} f > , \quad f \in \mathcal{E}(\mathbb{R})$$

is an isomorphism of $\mathcal{E}'(\mathbb{R})$ into itself, satisfying the intertwining relation

$$l_\beta(S^*_{\alpha,\beta} T) = S^*_{\alpha,\beta}(l_\alpha T) , \quad T \in \mathcal{E}'(\mathbb{R}) \tag{23}$$

Proof. From theorem 3.2, we deduce by duality that $S^*_{\alpha,\beta}$ is an isomorphism from $\mathcal{E}'(\mathbb{R})$ into itself. Using theorem 3.1, we obtain

$$< l_\beta S^*_{\alpha,\beta} T, f > = < S^*_{\alpha,\beta} l_\alpha T, f >$$

which gives the wanted result.

□

Remark 4.1 *From theorem 3.2 we deduce that*

$$S^{*}_{\alpha,\beta} = (\chi_\beta^{-1})^* \circ \chi_\alpha^* \tag{24}$$

Definition 4.2 *We define the Bessel-Struve transform on $\mathcal{E}'(\mathbb{R})$ by*

$$\forall\, T \in \mathcal{E}'(\mathbb{R})\,, \forall\, \lambda \in \mathbb{R},\;\; \mathcal{F}^{\alpha}_{B,S}(T)(\lambda) = < T, \Phi_\alpha(-i\lambda.) > \tag{25}$$

Proposition 4.1 *[1, Proposition 4.4] For all $T \in \mathcal{E}'(\mathbb{R})$,*

$$\mathcal{F}^{\alpha}_{B,S}(T) \;=\; \mathcal{F} \circ \chi^{\star}_{\alpha}(T) \tag{26}$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathcal{E}'(\mathbb{R})$.

The Bessel-Struve translation operator is given by

$$\tau_a f(x) = \chi_{\alpha,a}\chi_{\alpha,x}[\chi_\alpha^{-1}(f)(a+x)]$$

It is shown in [2, Proposition 4] that this translation operator is the unique solution C^∞ on $\mathbb{R}\times\mathbb{R}$ of the Cauchy problem

$$\begin{cases} l_{\alpha,a}u(a,x) = l_{\alpha,x}u(a,x) \\ u(0,x) = f(x) \\ D_a u(0,x) = Df(x) \end{cases} \qquad \text{where } f \in \mathcal{E}(\mathbb{R})$$

This translation operator has the following properties

(i) The operator τ_a is a linear continuous operator from $\mathcal{E}(\mathbb{R})$ into itself.

(ii) $\forall\, u \in \mathcal{E}(\mathbb{R})$, $\forall a, x \in \mathbb{R}$, we have

$$\begin{array}{l} \tau_a u(x) = \tau_x u(a), \;\; \tau_0 u(x) = u(x) \\ \tau_a \circ \tau_x = \tau_x \circ \tau_a, \;\; l_\alpha \circ \tau_a = \tau_a \circ l_\alpha \end{array}$$

(iii) The operator τ_a verifies the following type product formula

$$\forall\, a, x \in \mathbb{R}, \qquad \tau_a(\Phi_\alpha(\lambda.))(x) = \Phi_\alpha(\lambda x)\,\Phi_\alpha(\lambda a) \tag{27}$$

Definition 4.3 *The convolution product of two elements T and K in $\mathcal{E}'(\mathbb{R})$ is defined by*

$$< T \star_\alpha K, f > = < T_a, < K_x, \tau_a f(x) >> \,, \quad f \in \mathcal{E}(\mathbb{R}) \tag{28}$$

The convolution product $\star_\alpha$ satisfies the following property

Proposition 4.2 *Let $T, K \in \mathcal{E}'(\mathbb{R})$ then*

$$\mathcal{F}^{\alpha}_{B,S}(T \star_\alpha K) = \mathcal{F}^{\alpha}_{B,S}(T)\, \mathcal{F}^{\alpha}_{B,S}(K) \tag{29}$$

Proof. From relations (25) and (28), we have

$$\mathcal{F}^{\alpha}_{BS}(T \star_\alpha K)(\lambda) =< T_x, < K_y, \tau_x \Phi_\alpha(\lambda y) >>$$

Invoking relations (27) and (25), we obtain the desired result. □

Theorem 4.2 *We have*

1. *For all $T \in \mathcal{E}'(\mathbb{R})$*
$$\mathcal{F}^{\alpha}_{BS}(T) = \mathcal{F}^{\beta}_{BS}(S^*_{\alpha,\beta} T)$$
2. *For all $T, K \in \mathcal{E}'(\mathbb{R})$,*
$$S^*_{\alpha,\beta}(T \star_\alpha K) = S^*_{\alpha,\beta}(T) \star_\beta S^*_{\alpha,\beta}(K)$$

Proof. The first statement can be deduced from relations (26) and (24). The second statement follows by applying $\mathcal{F}^{\beta}_{BS}$ on both sides and using statement 1 and relation (29). □

Definition 4.4 *For f continuous function on $\mathbb{R}$, with compact support, we define the Dual Sonine transform denoted ${}^tS_{\alpha,\beta}$ by*

$${}^tS_{\alpha,\beta}(f)(x) = a_{\alpha,\beta} \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha-\beta-1} y\, f(sgn(x)y)dy\,, \quad x \in \mathbb{R}^* \tag{30}$$

where $a_{\alpha,\beta}$ is given in Proposition 3.1.

Remark 4.2 *Invoking relations (30) and (11), we get*

$${}^tS_{\alpha,\beta}(f)(x) = \frac{\sqrt{\pi}\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+\frac{1}{2})} W_{\alpha-\beta-\frac{1}{2}}(f)(x)\,, \quad x \in \mathbb{R} \tag{31}$$

Proposition 4.3 *the Dual Sonine transform verifies the following relation for all $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$,*

$$\int_{\mathbb{R}} S_{\alpha,\beta}(g)(x)\, f(x)\, |x|^{2\alpha+1}\, dx \;=\; \int_{\mathbb{R}} {}^tS_{\alpha,\beta}(f)(x)\, g(x)\, |x|^{2\beta+1}\, dx \tag{32}$$

Proof. We obtain the result using relations (16) and (30) and Fubini's theorem. □

Theorem 4.3 *For all f in $\mathcal{D}(\mathbb{R})$, we have*

$$ {}^tS_{\alpha,\beta}(f) = V_\beta(W_\alpha(f)) \tag{33}$$

Proof. From relations (32), (21),(13) and (14) we obtain for f in $\mathcal{D}(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$

$$\int_{\mathbb{R}} {}^tS_{\alpha,\beta}(f)(y)g(y)|y|^{2\beta+1}dy = \int_{\mathbb{R}} V_\beta(W_\alpha(f))(y)g(y)|y|^{2\beta+1}dy$$

As the functions ${}^tS_{\alpha,\beta}(f)$ and $V_\beta(W_\alpha(f))$ are with compact support , then

$$ {}^tS_{\alpha,\beta}(f) = V_\beta(W_\alpha(f)) \qquad \text{a.e}$$

Since both functions are continuous on $\mathbb{R}^*$, we get
$\forall\, x \in \mathbb{R}^*, \quad {}^tS_{\alpha,\beta}(f)(x) = V_\beta(W_\alpha(f))(x)$ □

Definition 4.5 *The Bessel-Struve transform is defined on $L^1_\alpha(\mathbb{R})$ by*

$$\forall\, \lambda \in \mathbb{R}, \qquad \mathcal{F}^\alpha_{B,S}(f)(\lambda) = \int_{\mathbb{R}} f(x)\, \Phi_\alpha(-i\lambda x)\, |x|^{2\alpha+1}dx \tag{34}$$

Proposition 4.4 *Let f be a function in $L^1_\alpha(\mathbb{R})$ with bounded support, then we have*

$$\mathcal{F}^\alpha_{B,S}(f) = \mathcal{F} \circ W_\alpha(f) \tag{35}$$

where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x)e^{-i\lambda x}dx$$

Proof. We proceed in similar way as proposition 3.2 in [1]

□

Corollary 4.1 *For all $f \in \mathcal{D}(\mathbb{R})$, we have the following decomposition:*

$$\mathcal{F}^\alpha_{BS}(f) = \mathcal{F}^\beta_{BS} \circ {}^tS_{\alpha,\beta}(f)$$

Proof. Let $f \in \mathcal{D}(\mathbb{R})$ with support included in $[-a, a]$. Invoking relation (31), we can see that ${}^tS_{\alpha,\beta}(f)$ is continuous function on $\mathbb{R}^*$ with support included in $[-a, a]$ and verifying $\lim\limits_{\substack{x\to 0\\ x<0}} f(x)$ and $\lim\limits_{\substack{x\to 0\\ x>0}} f(x)$ exist. Then we deduce the result from relation (35), theorem 4.3 and Lemma 2.1. □

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